



Well-posedness without semicontinuity for parametric quasiequilibria and quasioptimization[☆]

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ABSTRACT

We consider quasiequilibrium and quasioptimization problems. A relaxed level closedness notion is proposed and used together with pseudocontinuity to establish sufficient conditions for parametric well-posedness and well-posedness without semicontinuity assumptions. We prove them in general formulations, though such relaxations allow us to improve some existing results even in simple cases of R^1 . Several new well-posedness results are also obtained. For topological settings we use sensitivity analysis while for problems on metric spaces we argue on diameters and Kuratowski's and Hausdorff's measures of noncompactness of approximate solution sets.

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1. Introduction

In their seminal papers, Hadamard [1] and Tikhonov [2] initiated two ways of developing a well-posedness study for various mathematical problems. For constrained optimization the pioneer work was [3] of Levitin and Polyak, who extended the definition for unconstrained problems in [2]. Observe that the notions of Hadamard and Tikhonov were proved closely related in [4,5]. Recently, these two notions have been more blended and linked to stability theory in parametric well-posedness study [6–12]. Well-posedness for various problems related to optimization has been recently intensively considered, see e.g.: for optimization problems [5,9,12–16], for variational inequalities [17–21], for Nash equilibria [22,23], for fixed-point problems [8,19,24], for inclusion problems [8,19,24] and for equilibrium problems [6,7,25]. In most cases it is commonly assumed at least that the involved functions are lower semicontinuous. But in many practical optimization and control problems we meet even nonsemicontinuous functions. In [9,26] a weaker notion of lower pseudocontinuity is introduced to investigate parametric constrained optimization. In this paper we propose generalized level closedness definitions and use them together with pseudocontinuity to consider well-posedness in the Tikhonov sense, which is more important in approximation study and numerical algorithms, because all algorithms consist of providing sequences of approximate solutions convergent to an exact one. Simple examples (e.g. Examples 2.1 and 2.2) ensure that these properties are properly weaker than semicontinuity and hence results under assumptions about these properties are significant in practical situations. Note that quasiequilibrium models contain quasivariational inequalities, complementarity problems, vector minimization problems, Nash equilibria, fixed-point and coincidence-point problems, traffic networks, etc. A quasioptimization problem is more general than an optimization one as constraint sets depend on the decision

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variable as well. This is a special case of a quasiequilibrium problem but we go into details due to its importance. We discuss well-posedness by tools of sensitivity analysis for general settings in topological spaces, since this property is closely related to stability, especially for parametric problems. When decision spaces are metric spaces, diameters and measures of noncompactness of approximate solution sets play a crucial role. Namely, well-posedness depends on whether these quantities tend to zero or not. We will be employing both Kuratowski's and Hausdorff's measures in this paper. Furthermore, in our results for optimization problems, a kind of marginal function participates as well. Since the solution existence of these problems have been intensively studied, we focus on well-posedness assuming always that solutions of the problem under consideration exist. Some of our results improve the counterparts in the recent papers [9,25]. The others are new. The results of the paper are followed by numerous examples explaining that all the assumptions we impose are already very relaxed and cannot be dropped.

In the rest of this section we state our problems and recall well-posedness notions. Section 2 is devoted to generalized level closedness and pseudocontinuity properties. In the next Section 3 we establish sufficient conditions for a quasiequilibrium problem to be parametrically well-posed. Section 4 contains well-posedness conditions for a quasioptimization problem.

Let X and Λ be Hausdorff topological spaces, $f : X \times X \times \Lambda \rightarrow R$ and $K_i : X \times \Lambda \rightarrow 2^X$, $i = 1, 2$. Our parametric quasiequilibrium problem consists of, for each $\lambda \in \Lambda$,

$$(\text{QEP}_\lambda) \quad \text{finding } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for all } y \in K_2(\bar{x}, \lambda), \quad f(\bar{x}, y, \lambda) \geq 0.$$

Let $g : X \times \Lambda \rightarrow \bar{R}$, where $\bar{R} = (-\infty, +\infty]$, and $K : X \times \Lambda \rightarrow 2^X$. Our parametric quasioptimization problem is, for each $\lambda \in \Lambda$,

$$(\text{QOP}_\lambda) \quad \begin{cases} \text{minimize} & g(x, \lambda) \\ \text{subject to} & x \in K(x, \lambda). \end{cases}$$

Instead of writing $\{(\text{QEP}_\lambda) : \lambda \in \Lambda\}$ for the family of quasiequilibrium problems, i.e. the parametric problem, we will simply write (QEP) in the sequel. (QOP) is defined similarly.

We first recall well-posedness notions.

Definition 1.1. Let $\{\lambda_n\}$ converge to $\bar{\lambda}$. For $x_n \in K_1(x_n, \lambda_n)$, the sequence $\{x_n\}$ is said to be an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$, if there exists a sequence $\{\varepsilon_n\}$ convergent to 0^+ such that, for all $y \in K_2(x_n, \lambda_n)$,

$$f(x_n, y, \lambda_n) + \varepsilon_n \geq 0.$$

Definition 1.2. Problem (QEP) is called well-posed at $\bar{\lambda}$ if

- (a) the solution set $S(\bar{\lambda})$ of $(\text{QEP}_{\bar{\lambda}})$ is nonempty;
- (b) for any sequence $\{\lambda_n\}$ convergent to $\bar{\lambda}$, every corresponding approximating sequence for (QEP) has a subsequence convergent to some point of $S(\bar{\lambda})$.

(QEP) is called uniquely well-posed at $\bar{\lambda}$ if $S(\bar{\lambda}) = \{\bar{x}\}$, a singleton, and every approximating sequence converges to \bar{x} . (QEP) (or any other problem) is called parametrically (uniquely) well-posed if it is (uniquely) well-posed at each $\lambda \in \Lambda$.

Definition 1.3. Let $\{\lambda_n\}$ converge to $\bar{\lambda}$ in Λ . For $x_n \in K(x_n, \lambda_n)$, the sequence $\{x_n\}$ is said to be an approximating (or minimizing) sequence for (QOP) corresponding to $\{\lambda_n\}$, if there exists a sequence $\{\varepsilon_n\}$ convergent to 0^+ such that

$$g(x_n, \lambda_n) \leq \inf_{x \in K(x_n, \lambda_n)} g(x, \lambda_n) + \varepsilon_n.$$

Definition 1.4. Problem (QOP) is called well-posed at $\bar{\lambda}$ if

- (a) $(\text{QOP}_{\bar{\lambda}})$ has solutions;
- (b) for any sequence $\{\lambda_n\}$ convergent to $\bar{\lambda}$, every corresponding approximating sequence for (QOP) has a subsequence convergent to some point of $S(\bar{\lambda})$.

We say that (QOP) is uniquely well-posed at $\bar{\lambda}$ if $S(\bar{\lambda}) = \{\bar{x}\}$, a singleton, and every approximating sequence converges to \bar{x} .

Note that, in the above definitions, like a number of authors, we require an approximating sequence to be (strictly) included in the constraint set, unlike the definition in [3].

2. Generalized level closedness and pseudocontinuity of functions

Let X be a topological space, $x_0 \in X$ and $f : X \rightarrow \bar{R}$. Recall that f is called sequentially upper (lower, respectively) semicontinuous, written shortly as usc (lsc, resp.), at x_0 if, for all sequences $\{x_n\}$ convergent to x_0 , $f(x_0) \geq \limsup f(x_n)$ ($f(x_0) \leq \liminf f(x_n)$).

($f(x_0) \leq \liminf f(x_n)$, resp.). Note that in this paper we are concerned always with sequential properties. Hence we write clearly “sequential” or “sequentially” only to remind the reader in case necessary. Observe that f is usc at x_0 if and only if for all $\{x_n\} \rightarrow x_0$ and all $b \in R$,

$$[f(x_n) \geq b, \forall n] \Rightarrow [f(x_0) \geq b]$$

and similarly for lower semicontinuity. Therefore, we propose the following natural definition.

Definition 2.1. Let X and Y be topological spaces, $f : X \rightarrow \bar{R}$ and $g : Y \rightarrow \bar{R}$.

(a) f is called (sequentially) upper 0-level closed with respect to (w.r.t.) g at $(x_0, y_0) \in X \times Y$ if, for any sequence $\{(x_n, y_n)\}$ convergent to (x_0, y_0) ,

$$[f(x_n) + g(y_n) \geq 0, \forall n] \Rightarrow [f(x_0) + g(y_0) \geq 0].$$

(b) f is called (sequentially) lower 0-level closed w.r.t. g at (x_0, y_0) if, for any sequence $\{(x_n, y_n)\}$ convergent to (x_0, y_0) ,

$$[f(x_n) + g(y_n) \leq 0, \forall n] \Rightarrow [f(x_0) + g(y_0) \leq 0].$$

If we have f in place of $f + g$ in the above inequalities, we say that f is upper (or lower) 0-level closed at x_0 . While if we have $b \in R$ instead of 0, then of course “0-level” is replaced by “ b -level”.

Remark 2.1. If f and g are usc (lsc, resp.) at x_0 and y_0 , respectively, then f is upper (lower, resp.) 0-level closed w.r.t. g at (x_0, y_0) . Indeed, if $\{(x_n, y_n)\} \rightarrow (x_0, y_0)$ and $f(x_n) + g(y_n) \geq 0$ for all n , one has

$$f(x_0) + g(y_0) \geq \limsup f(x_n) + \limsup g(y_n) \geq \limsup [f(x_n) + g(y_n)] \geq 0.$$

From now on we use id to denote the identity map on R_+ . The following example shows that the converse of the above remark is not true.

Example 2.1. Let $f : R \rightarrow R$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in Q, \\ 1, & \text{if } x \in R \setminus Q, \end{cases}$$

where Q is the set of the rational numbers. Then f is upper 0-level closed w.r.t. id at (x, y) , for all $(x, y) \in R \times R_+$, but f is neither usc at any $x \in Q$ nor lsc at any $x \in R \setminus Q$.

Definition 2.2 ([9,26]). Let X be a topological space and $f : X \rightarrow \bar{R}$.

(a) f is said to be (sequentially) upper pseudocontinuous at $x_0 \in X$ if,

$$[f(x) > f(x_0)] \Rightarrow [\text{for any } \{x_n\} \rightarrow x_0, f(x) > \limsup f(x_n)].$$

(b) f is called lower pseudocontinuous at $x_0 \in X$ if,

$$[f(x) < f(x_0)] \Rightarrow [\text{for any } \{x_n\} \rightarrow x_0, f(x) < \liminf f(x_n)].$$

(c) f is termed pseudocontinuous at $x_0 \in X$ if it is both lower and upper pseudocontinuous at this point.

The class of the upper pseudocontinuous functions strictly contains that of the usc functions, see [26]. We include here a new simple illustrative example.

Example 2.2. Let $f : R \rightarrow R$ be defined by

$$f(x) = \begin{cases} x + 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ x - 1, & \text{if } x < 0. \end{cases}$$

Then, f is pseudocontinuous at 0 but neither usc nor lsc at 0.

We note further that if f and g are lsc (or usc) at x_0 then $f + g$ is lsc (usc, resp.) at x_0 . Unfortunately, this property does not hold for pseudocontinuous functions as shown by

Example 2.3. Let $f_1, g_1 : R \rightarrow R$ be defined as follows

$$f_1(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ \frac{x}{2}, & \text{if } x < 0 \end{cases} \quad \text{and} \quad g_1(x) = -x.$$

Then, f_1 is lower pseudocontinuous at 0 and g_1 is continuous at 0. But

$$(f_1 + g_1)(x) = \begin{cases} -x + 1, & \text{if } x \geq 0, \\ -\frac{x}{2}, & \text{if } x < 0 \end{cases}$$

is not lower pseudocontinuous at 0.

To see the same situation for upper pseudocontinuity let

$$f_2(x) = \begin{cases} -1, & \text{if } x \geq 0, \\ -\frac{x}{2}, & \text{if } x < 0 \end{cases} \quad \text{and} \quad g_2(x) = x.$$

Then at 0, f_2 is upper pseudocontinuous and g_2 is continuous. However,

$$(f_2 + g_2)(x) = \begin{cases} x - 1, & \text{if } x \geq 0, \\ \frac{x}{2}, & \text{if } x < 0 \end{cases}$$

is not upper pseudocontinuous at 0.

Lemma 2.1 ([9, Proposition 2.3]). Let X be a topological space. Then $f : X \rightarrow \bar{R}$ is pseudocontinuous in X if and only if, for all sequences $\{x_n\}$ and $\{y_n\}$ in X , convergent to x and y , respectively,

$$[f(y) < f(x)] \Rightarrow [\limsup f(y_n) < \liminf f(x_n)].$$

3. Quasiequilibrium problem (QEP)

For well-posedness of (QEP) in general topological settings we need the following facts which are well known and often used in sensitivity analysis (see e.g. [27–30] and references therein).

Remark 3.1. Let $Q : X \rightarrow 2^Y$ be a multimap between two topological spaces. Then the following assertions hold.

- (i) If $Q(\bar{x})$ is compact, then Q is usc at \bar{x} if and only if for any sequence $\{x_n\}$ convergent to \bar{x} and $y_n \in Q(x_n)$, there is a subsequence $\{y_{n_k}\}$ convergent to some $y \in Q(\bar{x})$.
- (ii) If, in addition, $Q(\bar{x}) = \{\bar{y}\}$ is a singleton then the above limit point y must be \bar{y} and the whole $\{y_n\}$ converges to \bar{y} .

By $S(\lambda)$ we denote the solution set of (QEP_λ) . For positive ε , the ε -solution set of (QEP_λ) is defined by

$$\tilde{S}(\lambda, \varepsilon) = \{x \in K_1(x, \lambda) \mid f(x, y, \lambda) + \varepsilon \geq 0, \forall y \in K_2(x, \lambda)\}.$$

When X and Λ are metric spaces, for positive ζ and ε , we define the following set of approximate solutions of the family (QEP), allowing also the parametric to vary around the considered point,

$$\Pi(\bar{\lambda}, \zeta, \varepsilon) := \bigcup_{\lambda \in B(\bar{\lambda}, \zeta)} \tilde{S}(\lambda, \varepsilon),$$

where $B(\bar{\lambda}, \zeta)$ is the closed ball centered at $\bar{\lambda}$ and with radius ζ .

Theorem 3.1. Assume that

- (i) X is compact, K_1 is closed and K_2 is lsc in $X \times \{\bar{\lambda}\}$;
- (ii) f is upper 0-level closed w.r.t. id in $K_1(X, \bar{\lambda}) \times K_2(X, \bar{\lambda}) \times \{\bar{\lambda}\} \times \{0\}$.

Then (QEP) is well-posed at $\bar{\lambda}$. Furthermore, if $S(\bar{\lambda})$ is a singleton, then this problem is uniquely well-posed at $\bar{\lambda}$.

Proof. We first check that $\tilde{S}(\cdot, \cdot)$ is usc at $(\bar{\lambda}, 0)$. Suppose to the contrary the existence of an open superset U of $\tilde{S}(\bar{\lambda}, 0)$ such that for all $\{(\lambda_n, \varepsilon_n)\}$ convergent to $(\bar{\lambda}, 0)$ in $\Lambda \times R_+$, there is $x_n \in \tilde{S}(\lambda_n, \varepsilon_n)$ such that $x_n \notin U$, for all n . By the compactness of X one can assume that $\{x_n\}$ converges to some x_0 . Since K_1 is closed at $(x_0, \bar{\lambda})$, $x_0 \in K_1(x_0, \bar{\lambda})$. If $x_0 \notin \tilde{S}(\bar{\lambda}, 0) = S(\bar{\lambda})$, there is $y_0 \in K_2(x_0, \bar{\lambda})$ such that $f(x_0, y_0, \bar{\lambda}) < 0$. The lower semicontinuity of K_2 in turn shows the existence of $y_n \in K_2(x_n, \lambda_n)$ such that $\{y_n\} \rightarrow y_0$. As $x_n \in \tilde{S}(\lambda_n, \varepsilon_n)$, one has

$$f(x_n, y_n, \lambda_n) + \varepsilon_n \geq 0.$$

By the upper 0-level closedness w.r.t. id of f , we have $f(x_0, y_0, \bar{\lambda}) \geq 0$, which is a contradiction. Thus, $x_0 \in \tilde{S}(\bar{\lambda}, 0) \subseteq U$, which is another contradiction, since $x_n \notin U$, for all n . Hence, \tilde{S} is usc at $(\bar{\lambda}, 0)$.

Now we prove that $S(\bar{\lambda})$ is compact by checking its closedness. Let $x_n \in S(\bar{\lambda})$ converge to x_0 . If $x_0 \notin S(\bar{\lambda})$, there exists $y_0 \in K_2(x_0, \bar{\lambda})$ such that $f(x_0, y_0, \bar{\lambda}) < 0$. In light of the lower semicontinuity of K_2 there is $y_n \in K_2(x_n, \bar{\lambda})$ such that $\{y_n\} \rightarrow y_0$. For all n one has $f(x_n, y_n, \bar{\lambda}) \geq 0$ as $x_n \in S(\bar{\lambda})$. By assumption (ii), one has $f(x_0, y_0, \bar{\lambda}) \geq 0$, which is impossible. Therefore, $x_0 \in S(\bar{\lambda})$ and hence $S(\bar{\lambda})$ is compact. By Remark 3.1 we are done. \square

The assumptions of [Theorem 3.1](#) are essential as indicated in the following examples.

Example 3.1 (*The Compactness of X Cannot be Dropped*). Let $X = \mathbb{R}$, $\Lambda = \mathbb{R}_+$, $K_1(x, \lambda) = K_2(x, \lambda) = \mathbb{R}$, $\bar{\lambda} = 0$ and $f(x, y, \lambda) = 2^{x-y} + \lambda$. It is clear that in $X \times \Lambda$, K_1 is closed and K_2 is lsc. (ii) holds as f is continuous in $X \times X \times \Lambda$. But $S(0) = \mathbb{R}$ for all $\lambda \in \Lambda$. Hence, (QEP) is not well-posed at 0. Indeed, let $\lambda_n = \frac{1}{n} \rightarrow 0$ and $x_n = n \in S(\bar{\lambda}_n)$ for all n . It is clear that $\{x_n\}$ has no convergent subsequence. The reason is that X is not compact.

Example 3.2 (*The Closedness of K_1 is Essential*). Let $X = [-2, 1]$, $\Lambda = [0, 1]$, $K_1(x, \lambda) = (-2\lambda, 1]$, $K_2(x, \lambda) = [0, 1]$, $\bar{\lambda} = 0$ and $f(x, y, \lambda) = x(x - y)$. It is not hard to see that X is compact, K_2 is lsc in $X \times \Lambda$, (ii) is fulfilled (by the continuity of f). But $S(0) = \{1\}$ and $S(\lambda) = \{0, 1\}$ for all $\lambda \in (0, 1]$. Therefore, (QEP) is not well-posed at 0. The reason is that K_1 is not closed at $X \times \{0\}$. Indeed, let $x_n = \lambda_n = \frac{1}{n}$ and $z_n = -\frac{1}{n} \in K_1(x_n, \lambda_n) = (-\frac{2}{n}, 1]$. We see that $\{z_n\}$ tends to 0 $\notin K_1(0, 0)$.

Example 3.3 (*The Lower Semicontinuity of K_2 Cannot be Dispensed*). Let $X = [-1, 1]$, $\Lambda = [0, 1]$, $K_1(x, \lambda) = [0, 1]$, $f(x, y, \lambda) = x + y$, $\bar{\lambda} = 0$ and

$$K_2(x, \lambda) = \begin{cases} \{-1, 0, 1\}, & \text{if } \lambda = 0, \\ \{0, 1\}, & \text{otherwise.} \end{cases}$$

Then X is compact, K_1 is closed in $X \times \Lambda$ and (ii) holds (by the continuity of f in $X \times X \times \Lambda$). But $S(0) = \{1\}$ and $S(\lambda) = \{0, 1\}$ for all $\lambda \in (0, 1]$. Thus, (QEP) is not well-posed at 0. The reason is that K_2 is not lsc in $X \times \{\bar{\lambda}\}$.

Example 3.4 ((ii) *Cannot be Dropped*). Let $X = [0, 1]$, $\Lambda = [0, 1]$, $K_1(x, \lambda) \equiv K_2(x, \lambda) = [0, 1]$ and

$$f(x, y, \lambda) = \begin{cases} x - y, & \text{if } \lambda = 0, \\ y - x, & \text{otherwise.} \end{cases}$$

It is clear that assumption (i) is satisfied and $S(0) = \{1\}$. Let $\lambda_n = \varepsilon_n = \frac{1}{n}$, and $x_n = 0 \in \tilde{S}(\lambda_n, \varepsilon_n)$. Then $\{x_n\}$ is an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$. But $\{x_n\} \rightarrow 0 \notin S(0)$ and hence $\{(QEP)_\lambda : \lambda \in \Lambda\}$ is not well-posed at $\bar{\lambda} = 0$. The reason is that assumption (ii) is violated. Indeed, taking $x_n = 0$, $y_n = 1$, $\lambda_n = \frac{1}{n}$ and $\varepsilon_n = 0$, we have $\{(x_n, y_n, \lambda_n, \varepsilon_n)\} \rightarrow (0, 1, 0, 0)$ and $f(x_n, y_n, \lambda_n) + \varepsilon_n = f(0, 1, \frac{1}{n}) = 1 > 0$ but $f(0, 1, 0) = -1 < 0$.

Remark 3.2. In the special case where $K(x, \lambda) \equiv X$, it is not hard to check that the assumption (ii) for f can be reduced to the same condition for $f(\cdot, y, \cdot)$, for all $y \in X$. Therefore, [Theorem 3.1](#) improves [Theorem 3.3](#) in [25]. Indeed, it suffices to check assumption (ii) of [Theorem 3.1](#) from the (assumed in [25]) monotonicity of $f(\cdot, \cdot, \bar{\lambda})$ and lower semicontinuity of $f(x, \cdot, \cdot)$. If $\{(x_n, \lambda_n)\} \rightarrow (x, \bar{\lambda})$ and $\{\varepsilon_n\}$ tends to 0^+ are such that

$$f(x_n, y, \bar{\lambda}_n) + \varepsilon_n \geq 0,$$

then, by the monotonicity, the inequalities

$$f(y, x, \bar{\lambda}) \leq \liminf f(y, x_n, \lambda_n) \leq \liminf (-f(x_n, y, \lambda_n)) \leq \liminf \varepsilon_n = 0$$

imply that $f(x, y, \bar{\lambda}) \geq 0$. Note further that we omit the hemicontinuity of $f(\cdot, \cdot, \bar{\lambda})$ and convexity of $f(x, \cdot, \bar{\lambda})$ imposed in [25].

Theorem 3.2. Let X and Λ be metric spaces.

- (i) If (QEP) is uniquely well-posed at $\bar{\lambda}$, then $\text{diam } \Pi(\bar{\lambda}, \zeta, \varepsilon) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.
- (ii) Conversely, if X is complete and the following conditions hold
 - (a) K_1 is closed and K_2 is lsc in $X \times \{\bar{\lambda}\}$;
 - (b) f is upper 0-level closed w.r.t. id in $K_1(X, \bar{\lambda}) \times K_2(X, \bar{\lambda}) \times \{\bar{\lambda}\} \times \{0\}$,
 then (QEP) is uniquely well-posed at $\bar{\lambda}$, provided that $\text{diam } \Pi(\bar{\lambda}, \zeta, \varepsilon) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.

Proof. (i) Suppose (QEP) is uniquely well-posed at $\bar{\lambda}$, but there is $\{(\zeta_n, \varepsilon_n)\} \rightarrow (0^+, 0^+)$ such that there are $n_0 \in \mathcal{N}$ (the set of natural numbers) and $r > 0$ such that, for all $n \geq n_0$,

$$\text{diam } \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n) > r.$$

Then, there exist $x_n^1, x_n^2 \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ such that $d(x_n^1, x_n^2) > \frac{r}{2}$. Consequently, there are $\lambda_n^1, \lambda_n^2 \in B(\bar{\lambda}, \zeta_n)$ such that

$$f(x_n^1, y, \lambda_n^1) + \varepsilon_n \geq 0, \quad \forall y \in K(x_n^1, \lambda_n^1)$$

and

$$f(x_n^2, y, \lambda_n^2) + \varepsilon_n \geq 0, \quad \forall y \in K(x_n^2, \lambda_n^2),$$

i.e. $\{x_n^1\}$ and $\{x_n^2\}$ are approximating sequences for (QEP) corresponding to $\{\lambda_n^1\}$ and $\{\lambda_n^2\}$, respectively. Hence, $\{x_n^1\}$ and $\{x_n^2\}$ converge to the unique solution of $(\text{QEP}_{\tilde{\lambda}})$, contradicting the fact that $d(x_n^1, x_n^2) > \frac{\varepsilon}{2} > 0$, for all n .

(ii) Let $\{\lambda_n\} \rightarrow \tilde{\lambda}$ and $\{x_n\}$ be an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$. Then there is $\{\varepsilon_n\} \rightarrow 0^+$ such that, for all $y \in K_2(x_n, \lambda_n)$ and all $n \in \mathcal{N}$,

$$f(x_n, y, \lambda_n) + \varepsilon_n \geq 0.$$

Consequently, x_n belongs to $\Pi(\tilde{\lambda}, \zeta_n, \varepsilon_n)$ with $\{\zeta_n\} := \{d(\lambda_n, \tilde{\lambda})\} \rightarrow 0^+$ as $n \rightarrow +\infty$. Since $\text{diam } \Pi(\tilde{\lambda}, \zeta_n, \varepsilon_n) \rightarrow 0^+$, $\{x_n\}$ is a Cauchy sequence and converges to some \bar{x} . By the closedness of K_1 at $(\bar{x}, \tilde{\lambda})$, $\bar{x} \in K(\bar{x}, \tilde{\lambda})$. Using the same argument as for Theorem 3.1, we deduce that $\bar{x} \in S(\tilde{\lambda})$. To complete the proof one shows that $(\text{QEP}_{\tilde{\lambda}})$ has a unique solution. If $S(\tilde{\lambda})$ has two distinct solutions \bar{x}_1 and \bar{x}_2 , it is not hard to see that \bar{x}_1 and \bar{x}_2 belong to $\Pi(\tilde{\lambda}, \zeta, \varepsilon)$, for all positive ζ and ε . It follows that

$$0 < d(\bar{x}_1, \bar{x}_2) \leq \text{diam} \Pi(\tilde{\lambda}, \zeta, \varepsilon),$$

which is impossible. \square

Remark 3.3. If $K(x, \lambda) \equiv X$, with the same argument as in Remark 3.2, we see that Theorem 3.2 improves Theorem 3.1 of [25]. Here we omit the hemicontinuity of $f(\cdot, \cdot, \tilde{\lambda})$ and convexity of $f(x, \cdot, \tilde{\lambda})$, which are required in that theorem.

The following example shows that we cannot replace the assumed unique well-posedness in Theorem 3.2(i) by well-posedness.

Example 3.5. Let $X = \Lambda = [0, 1]$, $K_1(x, \lambda) \equiv K_2(x, \lambda) = [0, 1]$ and $f(x, y, \lambda) = 1$. Then (QEP) is well-posed in Λ . But $\Pi(\lambda, \zeta, \varepsilon) = [0, 1]$ and hence its diameter does not converge to 0.

In the sequel we will need the following notions of measures of noncompactness.

Definition 3.1. Let M be a nonempty subset of a metric space X .

(i) The Kuratowski measure of M is

$$\mu(M) = \inf \left\{ \varepsilon > 0 \mid M \subseteq \bigcup_{k=1}^n M_k \text{ and } \text{diam } M_k \leq \varepsilon, k = 1, \dots, n, \text{ for some } n \in \mathcal{N} \right\}.$$

(ii) The Hausdorff measure of M is

$$\eta(M) = \inf \left\{ \varepsilon > 0 \mid M \subseteq \bigcup_{k=1}^n B(x_k, \varepsilon), x_k \in X, \text{ for some } n \in \mathcal{N} \right\}.$$

The following inequalities are obtained in [31]

$$\eta(M) \leq \mu(M) \leq 2\eta(M).$$

The measures μ and η share many properties and we will use γ in the sequel to denote either one of them. γ is a regular measure (see [32,33]), i.e. it enjoys the following properties

- (a) $\gamma(M) = +\infty$ if and only if the set M is unbounded;
- (b) $\gamma(M) = \gamma(\text{cl} M)$;
- (c) from $\gamma(M) = 0$ it follows that M is a totally bounded set;
- (d) if X is a complete space and if $\{A_n\}$ is a sequence of closed subsets of X such that $A_{n+1} \subseteq A_n$ for each $n \in \mathcal{N}$ and $\lim_{n \rightarrow +\infty} \gamma(A_n) = 0$, then $K := \bigcap_{n \in \mathcal{N}} A_n$ is a nonempty compact set and $\lim_{n \rightarrow +\infty} H(A_n, K) = 0^+$, where H is the Hausdorff metric;
- (e) from $M \subseteq N$ it follows that $\gamma(M) \leq \gamma(N)$.

Theorem 3.3.

- (i) If (QEP) is well-posed at $\tilde{\lambda}$, then $\gamma(\Pi(\tilde{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.
- (ii) Conversely, if X is complete, Λ is compact or finite dimensional and the following conditions hold
 - (a) K_1 is closed and K_2 is lsc in $X \times \Lambda$;
 - (b) f is upper b -level closed in $K_1(X, \Lambda) \times K_2(X, \Lambda) \times \Lambda$, for all $b < 0$,
 then (QEP) is well-posed at $\tilde{\lambda}$, provided that $\gamma(\Pi(\tilde{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.

Proof. Let γ be the Hausdorff measure η (for the Kuratowski measure case the argument is similar).

(i) Assume that (QEP) is well-posed at $\bar{\lambda}$ and $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$. Since $S(\bar{\lambda}) \subseteq \Pi(\bar{\lambda}, \zeta, \varepsilon)$ for all $\zeta, \varepsilon > 0$,

$$H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) = H^*(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})),$$

where $H^*(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{b \in B} d(a, b)$. Let $\{x_n\}$ be any sequence in $S(\bar{\lambda})$. Since $\{x_n\}$ is an approximating sequence for (QEP), there is a subsequence convergent to some point of $S(\bar{\lambda})$. Hence, $S(\bar{\lambda})$ is compact.

If $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^n B(z_k, \varepsilon)$, then

$$\Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \bigcup_{k=1}^n B(z_k, \varepsilon + H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})))$$

and hence

$$\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) + \gamma(S(\bar{\lambda})).$$

Since $S(\bar{\lambda})$ is compact, $\eta(S(\bar{\lambda})) = 0$. So we have

$$\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})).$$

Now we claim that $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$. Indeed, suppose to the contrary that there are $\rho > 0$, $\{(\zeta_n, \varepsilon_n)\} \rightarrow (0^+, 0^+)$ and $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ such that, for all $n \in \mathcal{N}$, $d(x_n, S(\bar{\lambda})) \geq \rho$. Since $\{x_n\}$ is an approximating sequence for (QEP), there is a subsequence convergent to some point of $S(\bar{\lambda})$, a contradiction.

(ii) Assume that $\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$. We first prove that $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed for all positive ζ and ε . Let $x_n \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$ be such that $\{x_n\} \rightarrow x$. Then, for each $n \in \mathcal{N}$, there is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that, for all $y \in K_2(x_n, \lambda_n)$,

$$f(x_n, y, \lambda_n) + \varepsilon \geq 0.$$

Since $B(\bar{\lambda}, \zeta)$ is compact, we can assume that $\{\lambda_n\} \rightarrow \lambda$ for some $\lambda \in B(\bar{\lambda}, \zeta)$. By the closedness of K_1 at (x, λ) , $x \in K_1(x, \lambda)$. We claim that, for all $y \in K_2(x, \lambda)$,

$$f(x, y, \lambda) + \varepsilon \geq 0.$$

Indeed, if there exists $y \in K_2(x, \lambda)$ such that $f(x, y, \lambda) + \varepsilon < 0$, there is $y_n \in K_2(x_n, \lambda_n)$ such that $\{y_n\} \rightarrow y$ as K_2 is lsc at (x, λ) . By the upper $-\varepsilon$ -level closedness of f at (x, y, λ) , there is $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0$, $f(x_n, y_n, \lambda_n) < -\varepsilon$, a contradiction. Since $\lambda \in B(\bar{\lambda}, \zeta)$, we have $x \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$. Hence, $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed.

Now we show that $S(\bar{\lambda}) = \bigcap_{\zeta > 0, \varepsilon > 0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$. We first check that $\bigcap_{\zeta > 0} \Pi(\bar{\lambda}, \zeta, \varepsilon) = \tilde{S}(\bar{\lambda}, \varepsilon)$. Indeed, it is easy to see that $\bigcap_{\zeta > 0} \Pi(\bar{\lambda}, \zeta, \varepsilon) \supseteq \tilde{S}(\bar{\lambda}, \varepsilon)$. Let $x \in \bigcap_{\zeta > 0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$. There is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that, for all $y \in K_2(x, \lambda_n)$, $f(x, y, \lambda_n) + \varepsilon \geq 0$. Since $x \in K_1(x, \lambda_n)$, $\{\lambda_n\} \rightarrow \bar{\lambda}$ and K_1 is closed, one sees that $x \in K_1(x, \bar{\lambda})$. Now we verify that $x \in \tilde{S}(\bar{\lambda}, \varepsilon)$. Indeed, for each $y \in K_2(x, \bar{\lambda})$, since K_2 is lsc at $(x, \bar{\lambda})$, there exists $y_n \in K_2(x, \lambda_n)$ with $\{y_n\} \rightarrow y$. Since $x \in S(\lambda_n, \varepsilon)$,

$$f(x, y_n, \lambda_n) + \varepsilon \geq 0.$$

By the upper $-\varepsilon$ -level closedness of f , one has

$$f(x, y, \bar{\lambda}) + \varepsilon \geq 0,$$

i.e. $\bigcap_{\zeta > 0} \Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \tilde{S}(\bar{\lambda}, \varepsilon)$. Hence, $\bigcap_{\zeta > 0} \Pi(\bar{\lambda}, \zeta, \varepsilon) = \tilde{S}(\bar{\lambda}, \varepsilon)$. Next, we have $S(\bar{\lambda}) = \bigcap_{\varepsilon > 0} \tilde{S}(\bar{\lambda}, \varepsilon) = \bigcap_{\zeta > 0, \varepsilon > 0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$.

Since $\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$, the regular measure properties of η imply that $S(\bar{\lambda})$ is compact and $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.

Let x_n be an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$, where $\{\lambda_n\} \rightarrow \bar{\lambda}$. There is $\{\varepsilon_n\} \rightarrow 0^+$ such that, for all $y \in K_2(x_n, \lambda_n)$ and all $n \in \mathcal{N}$,

$$f(x_n, y, \lambda_n) + \varepsilon_n \geq 0.$$

This means that $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ with $\zeta_n := d(\bar{\lambda}, \lambda_n)$. We see that

$$d(x_n, S(\bar{\lambda})) \leq H(\Pi(\bar{\lambda}, \zeta_n, \varepsilon_n), S(\bar{\lambda})) \rightarrow 0^+.$$

Hence, there is $\bar{x}_n \in S(\bar{\lambda})$ such that $d(x_n, \bar{x}_n) \rightarrow 0$ as $n \rightarrow \infty$. By the compactness of $S(\bar{\lambda})$, there is a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ convergent to some point \bar{x} of $S(\bar{\lambda})$. Therefore, the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ tends to \bar{x} . Hence, (QEP) is well-posed at $\bar{\lambda}$. \square

The following examples show that the assumptions of Theorem 3.3(ii) are essential.

Example 3.6 (The Closedness of K_1 cannot be Dispensed). Let $X = \mathbb{R}$, $\Lambda = [0, 1]$, $K_1(x, \lambda) = (-\lambda, 1]$, $K_2(x, \lambda) \equiv [0, 1]$, $f(x, y, \lambda) = x(x - y)$ and $\bar{\lambda} = 0$. It is easy to see that X is complete, Λ is compact, K_2 is lsc in $X \times \Lambda$. Condition (ii)(b) holds

since f is continuous in $X \times X \times \Lambda$. Moreover, $\Pi(0, \zeta, \varepsilon) \subseteq [-1, 1]$ and hence $\gamma(\Pi(0, \zeta, \varepsilon)) = 0$. But $S(0) = \{1\}$ and $S(\lambda) = \{0, 1\}$ for all $\lambda \in (0, 1]$. Hence, (QEP) is not well-posed at 0. The reason is that K_1 is not closed at $(0, 0)$. Indeed, let $x_n = \lambda_n = \frac{1}{n}$ and $z_n = \frac{1}{n} \in K_1(x_n, \lambda_n)$. We see that $z_n \rightarrow 0 \notin K_1(0, 0)$, and hence K_1 is not closed at $(0, 0)$.

Example 3.7 (The Lower Semicontinuity of K_2 is Essential). Let X , Λ and $\bar{\lambda}$ be as in Example 3.6, $K_1(x, \lambda) = [0, 1]$, $f(x, y, \lambda) = x + y$ and

$$K_2(x, \lambda) = \begin{cases} \{-1, 0, 1\}, & \text{if } \lambda = 0, \\ \{0, 1\}, & \text{otherwise.} \end{cases}$$

It is not hard to see that X is complete, Λ is compact, K_1 is closed in $X \times \Lambda$. (ii)(b) is satisfied as f is continuous in $X \times X \times \Lambda$. $\Pi(0, \zeta, \varepsilon) \subseteq [0, 1]$ and hence $\gamma(\Pi(0, \zeta, \varepsilon)) = 0$. But $S(0) = \{1\}$, $S(\lambda) = \{0, 1\}$ for all $\lambda \in (0, 1]$. Thus, (QEP) is not well-posed at 0. The reason is that K_2 is not lsc in $X \times \Lambda$.

Example 3.8 (Condition (ii)(b) cannot be dropped). Let X , Λ , K_1 , $\bar{\lambda}$ be as in Example 3.7, $K_2(x, \lambda) = \{\lambda, 1 + \lambda\}$ and

$$f(x, y, \lambda) = \begin{cases} -1, & \text{if } x + y = 1, \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that X is complete, Λ is compact, (ii)(a) holds and $\gamma(\Pi(0, \zeta, \varepsilon)) = 0$. But $S(0) = (0, 1)$, $S(\lambda) = [0, 1]$ for all $\lambda \in (0, 1]$. Therefore, (QEP) is not well-posed at 0. The reason is that assumption (ii)(b) is violated. Indeed, let $(x_n, y_n, \lambda_n) = (\frac{1}{n}, 1 - 2\frac{1}{n}, \frac{1}{n})$. We see that

$$f(x_n, y_n, \lambda_n) = 1 \geq -\frac{1}{2}.$$

But $\{(x_n, y_n, \lambda_n)\} \rightarrow (0, 1, 0)$ and

$$f(0, 1, 0) = -1 \not\geq -\frac{1}{2}.$$

Remark 3.4. In the special case where $K_1(x, \lambda) = K_2(x, \lambda) \equiv X$, it is easy to see that assumption (ii)(b) of Theorem 3.3 can be reduced to the corresponding one of $f(\cdot, y, \lambda)$, for all $y \in X$. Theorem 3.2 of [25] has the same conclusion as Theorem 3.3 (for this particular case), but only for the Kuratowski measure μ . Observe that the upper semicontinuity of $f(\cdot, y, \cdot)$, required in that theorem, implies the upper b -level closedness of $f(\cdot, y, \cdot)$ for all $b < 0$ as imposed in Theorem 3.3. Note further that (see Proposition 2.1 of [6]) the upper semicontinuity of $f(\cdot, y, \cdot)$ is equivalent to the upper b -level closedness of $f(\cdot, y, \cdot)$ for all b .

The following example gives a case where Theorem 3.3 is easy to be employed, but Theorem 3.2 of [25] does not work.

Example 3.9. Let $X = \Lambda = [0, 1]$, $K_1(x, \lambda) = K_2(x, \lambda) = [0, 1]$, $\bar{\lambda} = 0$ and

$$f(x, y, \lambda) = \begin{cases} 0, & \text{if } \lambda \in [0, 1] \cap \mathbb{Q}, \\ 1, & \text{if } \lambda \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q}). \end{cases}$$

Then the assumptions in (ii) of Theorem 3.3 are satisfied, and hence this theorem yields the well-posedness of (QEP) at 0. (In fact, $S(\lambda) = [0, 1]$ for all $\lambda \in [0, 1]$.) But $f(\cdot, y, \cdot)$ is not usc in $X \times \Lambda$, and hence Theorem 3.2 of [25] is not in use.

4. Quasioptimization problem (QOP)

We first investigate parametric well-posedness of this problem in topological settings.

Theorem 4.1. Assume that

- (i) X is compact and K is closed and lsc in $X \times \{\bar{\lambda}\}$;
- (ii) g is pseudocontinuous in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$.

Then (QOP) is well-posed at $\bar{\lambda}$. Furthermore, if (QOP) has a unique solution, this problem is uniquely well-posed at $\bar{\lambda}$.

Proof. By setting $K_1(x, \lambda) = K_2(x, \lambda) = K(x, \lambda)$, for all $(x, \lambda) \in X \times \Lambda$ and $f(x, y, \lambda) = g(y, \lambda) - g(x, \lambda)$, (QOP) becomes a special case of (QEP). To apply Theorem 3.1 we check its assumption (ii). Let x_n and y_n be in $K(X, \lambda_n)$ and $\varepsilon_n \in (0, +\infty)$ be such that $\{(x_n, y_n, \lambda_n, \varepsilon_n)\} \rightarrow (x, y, \bar{\lambda}, 0)$ and

$$f(x_n, y_n, \lambda_n) + \varepsilon_n \geq 0.$$

There are \bar{x}_n and \bar{y}_n in X such that $x_n \in K(\bar{x}_n, \lambda_n)$ and $y_n \in K(\bar{y}_n, \lambda_n)$. Due to the compactness of X one can assume that $\{\bar{x}_n\} \rightarrow \bar{x}$ and $\{\bar{y}_n\} \rightarrow \bar{y}$, for some $\bar{x}, \bar{y} \in X$. As K is closed in $X \times \{\bar{\lambda}\}$, we have $x \in K(\bar{x}, \bar{\lambda})$ and $y \in K(\bar{y}, \bar{\lambda})$.

Now suppose ad absurdum that $g(y, \bar{\lambda}) < g(x, \bar{\lambda})$. By Lemma 2.1 we have

$$\limsup g(y_n, \lambda_n) < \liminf g(x_n, \lambda_n).$$

Hence, there are $t_1, t_2 \in \mathbb{R}$ and $n_0 \in \mathcal{N}$ such that, for $n \geq n_0$,

$$g(y_n, \lambda_n) \leq t_1 < t_2 \leq g(x_n, \lambda_n)$$

and then

$$g(y_n, \lambda_n) - g(x_n, \lambda_n) \leq t_1 - t_2 < 0,$$

which is impossible and we are done. \square

Let $m : X \times \Lambda \rightarrow \mathbb{R}$ be the following kind of marginal functions

$$m(x, \lambda) := \inf\{g(y, \lambda) \mid y \in K(x, \lambda)\}.$$

When (QOP) is given on metric spaces, similarly as for (QEP) we define \tilde{S} and Π as follows

$$\tilde{S}(\lambda, \varepsilon) = \{x \in K(x, \lambda) \mid g(x, \lambda) \leq m(x, \lambda) + \varepsilon\},$$

$$\Pi(\bar{\lambda}, \zeta, \varepsilon) = \bigcup_{\lambda \in B(\bar{\lambda}, \zeta)} \tilde{S}(\lambda, \varepsilon).$$

Theorem 4.2. Assume that

- (i) X is compact and K is closed in $X \times \{\bar{\lambda}\}$;
- (ii) g is lower pseudocontinuous in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$;
- (iii) m is usc in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$.

Then (QOP) is well-posed at $\bar{\lambda}$. Furthermore, if (QOP) has a unique solution, it is uniquely well-posed at $\bar{\lambda}$.

Proof. We check first that \tilde{S} is usc at $(\bar{\lambda}, 0)$. Suppose to the contrary the existence of an open superset U of $\tilde{S}(\bar{\lambda}, 0)$ such that for all $\{(\lambda_n, \varepsilon_n)\}$ convergent to $(\bar{\lambda}, 0^+)$ in $\Lambda \times \mathbb{R}_+$, there is $x_n \in \tilde{S}(\lambda_n, \varepsilon_n)$ such that $x_n \notin U$, for all n . By the compactness of X one can assume that $\{x_n\}$ tends to some x_0 . Since K is closed at $(x_0, \bar{\lambda})$, $x_0 \in K(x_0, \bar{\lambda})$. If $x_0 \notin \tilde{S}(\bar{\lambda}, 0) = S(\bar{\lambda})$, there is $y_0 \in K(x_0, \bar{\lambda})$ such that $g(y_0, \bar{\lambda}) < g(x_0, \bar{\lambda})$. Since g is lower pseudocontinuous at $(x_0, \bar{\lambda})$, we have

$$m(x_0, \bar{\lambda}) \leq g(y_0, \bar{\lambda}) < \liminf g(x_n, \lambda_n).$$

The upper semicontinuity of m at $(x_0, \bar{\lambda})$ yields some $t \in \mathbb{R}$ such that

$$\limsup m(x_n, \lambda_n) < t < \liminf g(x_n, \lambda_n).$$

Hence, there is $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0$,

$$m(x_n, \lambda_n) - g(x_n, \lambda_n) < t - g(x_n, \lambda_n).$$

As $x_n \in \tilde{S}(\lambda_n, \varepsilon_n)$,

$$-\varepsilon_n \leq m(x_n, \lambda_n) - g(x_n, \lambda_n) \leq 0.$$

Therefore,

$$0 = \lim_{n \rightarrow +\infty} [m(x_n, \lambda_n) - g(x_n, \lambda_n)] \leq t - \liminf_{n \rightarrow +\infty} g(x_n, \lambda_n) < 0.$$

This contradiction shows that $x_0 \in S(\bar{\lambda})$. Then another contradiction is obtained as $x_n \notin U$. Thus, \tilde{S} is usc at $(\bar{\lambda}, 0)$. Now we prove that $S(\bar{\lambda})$ is compact by checking its closedness. Let $\{x_n\} \subseteq S(\bar{\lambda})$ converge to x_0 . As $S(\bar{\lambda}) \subseteq \tilde{S}(\bar{\lambda}, \varepsilon_n)$, by the preceding argument one sees that $x_0 \in S(\bar{\lambda})$. By Remark 3.1, (QOP) is well-posed at $\bar{\lambda}$. \square

The following examples explain that Theorems 4.1 and 4.2 are incomparable and each of them may be applicable in different situations.

Example 4.1. Let $X = \Lambda = [0, 1]$, $K(x, \lambda) = [0, 1]$, $\bar{\lambda} = 1$ and

$$g(x, \lambda) = \begin{cases} (1+x)(1-\lambda), & \text{if } 0 \leq \lambda < 1, \\ -1, & \text{if } \lambda = 1. \end{cases}$$

It is clear that K is continuous, X is compact and g is lower pseudocontinuous in $[0, 1] \times [0, 1]$. Now we check that g is upper pseudocontinuous at $(x, 1)$, for all $x \in [0, 1]$. Indeed, assume that $g(y, \lambda) > g(x, 1) = -1$ and $\{(x_n, \lambda_n)\} \rightarrow (x, 1)$. It is clear that, $g(y, \lambda) > 0$ as $\lambda < 1$ and $\limsup_{n \rightarrow +\infty} g(x_n, \lambda_n) = 0$. So $g(y, \lambda) > \limsup_{n \rightarrow +\infty} g(x_n, \lambda_n)$. Hence, the assumptions of Theorem 4.1 are satisfied and we obtain the well-posedness at 1 (in fact, $S(1) = [0, 1]$ and $S(\lambda) = \{0\}$ for all $0 \leq \lambda < 1$).

However,

$$m(x, \lambda) \equiv m(\lambda) = \begin{cases} 1 - \lambda, & \text{if } 0 \leq \lambda < 1, \\ -1, & \text{if } \lambda = 1 \end{cases}$$

is not usc at 1. Therefore, [Theorem 4.2](#) cannot be applied in this case.

Example 4.2. Let $X = \Lambda = [0, 1]$, $K(x, \lambda) = [0, 1]$, $\bar{\lambda} = 0$ and

$$g(x, \lambda) = \begin{cases} 0, & \text{if } \lambda = 0 \text{ and } 0 \leq x < 1, \\ \lambda(1 - x), & \text{if } 0 < \lambda \leq 1 \text{ and } 0 \leq x < 1, \\ -1, & \text{if } x = 1. \end{cases}$$

Then K is continuous and X is compact. g is lower pseudocontinuous at $(x, 0)$, for all $x \in [0, 1]$. Indeed, if $g(y, \lambda) < g(x, 0)$ then $x < 1$, and hence $g(x, 0) = 0$. So $g(y, \lambda) = -1$ and $y = 1$. If $\{(x_n, \lambda_n)\} \rightarrow (x, 0)$, there is $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0$, $x_n < 1$. So, we have $\liminf g(x_n, \lambda_n) = 0$. Thus, $g(y, \lambda) < \liminf g(x_n, \lambda_n)$, i.e., g is lower pseudocontinuous at $(x, 0)$. However, g is not upper pseudocontinuous in $[0, 1] \times \{0\}$. Indeed, let $y = \frac{1}{2}$ and $\lambda = 0$. Then

$$0 = g\left(\frac{1}{2}, 0\right) > g(1, 0) = -1.$$

Take $x_n = 1 - \frac{1}{n+1}$ and $\lambda_n = \frac{1}{n+1}$. Then $\{(x_n, \lambda_n)\} \rightarrow (1, 0)$ as $n \rightarrow +\infty$. It is easy to see that

$$\limsup g(x_n, \lambda_n) = \limsup \lambda_n(1 - x_n) = 0,$$

and hence $g\left(\frac{1}{2}, 0\right) \neq \limsup g(x_n, \lambda_n)$. Therefore, [Theorem 4.1](#) is not in use. Fortunately, the assumptions of [Theorem 4.2](#) are satisfied, since $m(x, \lambda) \equiv m(\lambda) = \inf_{x \in [0, 1]} g(x, \lambda) = -1$, for all $\lambda \in [0, 1]$ and hence m is continuous in $[0, 1]$. [Theorem 4.2](#) yields the well-posedness of (QOP) at 0 (in fact, $S(\lambda) = \{1\}$, for all $\lambda \in [0, 1]$).

Now we pass to well-posedness conditions in terms of the diameter of $\Pi(\lambda, \zeta, \varepsilon)$.

Theorem 4.3. Assume that X is a metric space.

- (i) If (QOP) is uniquely well-posed at $\bar{\lambda}$, then $\text{diam } \Pi(\bar{\lambda}, \zeta, \varepsilon) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.
- (ii) Conversely, assume that X is complete and the following conditions hold
 - (a) K is closed and lsc in $X \times \{\bar{\lambda}\}$;
 - (b) either of the following conditions holds
 - (b₁) g is pseudocontinuous in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$;
 - (b₂) in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$, g is lower pseudocontinuous and m is usc.

Then (QOP) is uniquely well-posed at $\bar{\lambda}$, provided that $\text{diam } \Pi(\bar{\lambda}, \zeta, \varepsilon) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.

Proof. (i) Suppose (QOP) is uniquely well-posed at $\bar{\lambda}$ but, for $\{(\zeta_n, \varepsilon_n)\} \rightarrow (0^+, 0^+)$, there are $n_0 \in \mathcal{N}$ and $r > 0$ such that, for all $n \geq n_0$, $\text{diam } \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n) > r$. Then, there exist $x_n^1, x_n^2 \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ such that $d(x_n^1, x_n^2) > \frac{r}{2}$. There are $\lambda_n^1, \lambda_n^2 \in B(\bar{\lambda}, \zeta_n)$ such that

$$g(x_n^1, \lambda_n^1) \leq m(x_n^1, \lambda_n^1) + \varepsilon_n$$

and

$$g(x_n^2, \lambda_n^2) \leq m(x_n^2, \lambda_n^2) + \varepsilon_n.$$

Since $\{x_n^1\}$ and $\{x_n^2\}$ are approximating sequences for (QOP) corresponding to $\{\lambda_n^1\}$ and $\{\lambda_n^2\}$, respectively, they converge to the unique solution and we obtain a contradiction.

(ii) Assume that $\{\lambda_n\} \rightarrow \bar{\lambda}$ and $\{x_n\}$ is an approximating sequence for (QOP) corresponding to $\{\lambda_n\}$. Then, there is $\{\varepsilon_n\} \rightarrow 0^+$ such that, for all $n \in \mathcal{N}$,

$$g(x_n, \lambda_n) \leq m(x_n, \lambda_n) + \varepsilon_n.$$

Hence x_n belongs to $\Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ with $\zeta_n := d(\lambda_n, \bar{\lambda})$. Since $\lim_{n \rightarrow +\infty} \text{diam } \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n) = 0^+$, $\{x_n\}$ is a Cauchy sequence and hence converges to some \bar{x} . The closedness of K_1 implies that $\bar{x} \in K(\bar{x}, \bar{\lambda})$. Using the same argument as for [Theorem 4.1](#) for the case (b₁) or [Theorem 4.2](#) for the case (b₂), we see that $\bar{x} \in S(\bar{\lambda})$. To complete the proof, we have to show that $(\text{QOP}_{\bar{\lambda}})$ has a unique solution. If $S(\bar{\lambda})$ has two distinct solutions \bar{x}_1 and \bar{x}_2 , they clearly belong to $\Pi(\bar{\lambda}, \zeta, \varepsilon)$, for all positive ζ and ε . This implies the contradiction that

$$0 < d(\bar{x}_1, \bar{x}_2) \leq \text{diam } \Pi(\bar{\lambda}, \zeta, \varepsilon). \quad \square$$

Theorem 4.4.

- (i) $\gamma(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$, if (QOP) is well-posed at $\bar{\lambda}$ (recall that γ is the Kuratowski measure or Hausdorff measure).

- (ii) Conversely, assume that X is complete and Λ is compact or finite dimensional. Impose further that,
- (a) K is closed in $X \times \Lambda$;
 - (b) g is lsc in $K(X, \Lambda) \times \Lambda$;
 - (c) m is usc in $K(X, \Lambda) \times \Lambda$.

Then (QOP) is well-posed at $\bar{\lambda}$, provided that $\gamma(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.

Proof. By similarity we discuss only the case where $\gamma = \mu$, the Kuratowski measure.

- (i) Assume that (QOP) is well-posed at $\bar{\lambda}$. Since, for all positive ζ and ε , $S(\bar{\lambda}) \subseteq \Pi(\bar{\lambda}, \zeta, \varepsilon)$, one has

$$H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) = H^*(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})).$$

Let $\{x_n\}$ be a sequence in $S(\bar{\lambda})$. Then $\{x_n\}$ is an approximating sequence for (QOP) and has a subsequence convergent to some point of $S(\bar{\lambda})$. Hence, $S(\bar{\lambda})$ is compact.

Let $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^n M_k$ with $\text{diam } M_k \leq \varepsilon$, for $k = 1, \dots, n$. Set

$$N_k = \{z \in X \mid d(z, M_k) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda}))\}.$$

We claim that

$$\Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \bigcup_{k=1}^n N_k.$$

Indeed, let $x \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$. Then $d(x, S(\bar{\lambda})) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda}))$. Since $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^n M_k$, we see that $d(x, \bigcup_{k=1}^n M_k) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda}))$. Hence, there is \bar{k} such that $d(x, M_{\bar{k}}) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda}))$, i.e. $x \in N_{\bar{k}}$. So, $\Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \bigcup_{k=1}^n N_k$. Note further that

$$\text{diam } N_k = \text{diam } M_k + 2H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \leq \varepsilon + 2H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})),$$

and hence, as $\mu(S(\bar{\lambda})) = 0$,

$$\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \leq 2H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) + \mu(S(\bar{\lambda})) = 2H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})).$$

Now we prove that $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$. Suppose to the contrary that there are $\rho > 0$, $\{(\zeta_n, \varepsilon_n)\} \rightarrow (0^+, 0^+)$ and $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ such that, for all $n \in \mathcal{N}$, $d(x_n, S(\bar{\lambda})) \geq \rho$. Since $\{x_n\}$ is an approximating sequence for (QOP), it has a subsequence convergent to some point of $S(\bar{\lambda})$, a contradiction. Therefore, $\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.

(ii) Assume that $\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$. We first show that $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed for all positive ζ and ε . Let $x_n \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$ and $\{x_n\} \rightarrow x$. Then, for each $n \in \mathcal{N}$, there is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that

$$g(x_n, \lambda_n) \leq m(x_n, \lambda_n) + \varepsilon.$$

Because $B(\bar{\lambda}, \zeta)$ is compact, we assume that $\{\lambda_n\} \rightarrow \lambda$ for some $\lambda \in B(\bar{\lambda}, \zeta)$. Since K is closed at (x, λ) , $x \in K(x, \lambda)$. By the lower semicontinuity of g and the upper semicontinuity of m at (x, λ) , we have

$$g(x, \lambda) \leq m(x, \lambda) + \varepsilon.$$

As $\lambda \in B(\bar{\lambda}, \zeta)$ we have $x \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$. Hence, $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed. Note further that $S(\bar{\lambda}) = \bigcap_{\zeta>0, \varepsilon>0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$ and $\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$. From the properties of μ it follows that $S(\bar{\lambda})$ is compact and $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \rightarrow 0^+$. Let $\{x_n\}$ be an approximating sequence for (QOP) corresponding to $\{\lambda_n\}$, where $\{\lambda_n\} \rightarrow \bar{\lambda}$. There is $\{\varepsilon_n\} \rightarrow 0^+$ such that, for all $n \in \mathcal{N}$,

$$g(x_n, \lambda_n) \leq m(x_n, \lambda_n) + \varepsilon_n.$$

Consequently, $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ with $\zeta_n := d(\bar{\lambda}, \lambda_n)$. We see that

$$d(x_n, S(\bar{\lambda})) \leq H(\Pi(\bar{\lambda}, \zeta_n, \varepsilon_n), S(\bar{\lambda})) \rightarrow 0^+.$$

By the compactness of $S(\bar{\lambda})$, there is a subsequence of $\{x_n\}$ converging to some point of $S(\bar{\lambda})$. Hence, (QOP) is well-posed at $\bar{\lambda}$. \square

Theorem 4.5. Assume that X is complete and Λ is compact or finite dimensional. Let the following conditions hold

- (a) K is closed and lsc in $X \times \Lambda$;
- (b) g is continuous in $K(X, \Lambda) \times \Lambda$.

Then (QOP) is well-posed at $\bar{\lambda}$, provided that $\gamma(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$.

Proof. We consider only the case $\gamma = \mu$. Let $\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \rightarrow 0^+$ as $(\zeta, \varepsilon) \rightarrow (0^+, 0^+)$. We prove that $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed for all positive ζ and ε . Let $x_n \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$ and $\{x_n\} \rightarrow x$. Then, for each $n \in \mathcal{N}$, there is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that

$$g(x_n, \lambda_n) \leq m(x_n, \lambda_n) + \varepsilon.$$

As $B(\bar{\lambda}, \zeta)$ is compact, we assume that $\{\lambda_n\} \rightarrow \lambda$ for some $\lambda \in B(\bar{\lambda}, \zeta)$. Then $x \in K(x, \lambda)$ as K is closed at (x, λ) . Now we show that,

$$g(x, \lambda) \leq m(x, \lambda) + \varepsilon.$$

By the lower semicontinuity of g at (x, λ) we have

$$g(x, \lambda) \leq \liminf g(x_n, \lambda_n) \leq \liminf m(x_n, \lambda_n) + \varepsilon.$$

Hence, it is sufficient to check that

$$\liminf m(x_n, \lambda_n) \leq m(x, \lambda),$$

that is

$$\liminf \inf_{y \in K(x_n, \lambda_n)} g(y, \lambda_n) \leq \inf_{y \in K(x, \lambda)} g(y, \lambda).$$

Suppose to the contrary the existence of $\delta > 0$ such that

$$\liminf \inf_{y \in K(x_n, \lambda_n)} g(y, \lambda_n) = \inf_{y \in K(x, \lambda)} g(y, \lambda) + \delta.$$

Then, there is $y_0 \in K(x, \lambda)$ such that

$$\liminf \inf_{y \in K(x_n, \lambda_n)} g(y, \lambda_n) > g(y_0, \lambda) + \frac{\delta}{2}.$$

Since K is lsc at (x, λ) , there is $y_n \in K(x_n, \lambda_n)$ such that $\{y_n\} \rightarrow y_0$. Taking into account the upper semicontinuity of g at (y_0, λ) , one has

$$g(y_0, \lambda) \geq \limsup g(y_n, \lambda_n) \geq \liminf \inf_{y \in K(x_n, \lambda_n)} g(y, \lambda_n) > g(y_0, \lambda) + \frac{\delta}{2},$$

which is a contradiction. Therefore, as $\lambda \in B(\bar{\lambda}, \zeta)$, we have $x \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$. Hence, $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed. The further argument is the same as the last part of the proof of [Theorem 4.4](#). \square

[Examples 4.1](#) and [4.2](#) show also that [Theorems 4.4](#) and [4.5](#) are incomparable.

Remark 4.1. In the special case where $K(x, \lambda) \equiv K(\lambda)$, i.e. (QOP) becomes an optimization problem, [Theorems 4.1–4.3](#) collapse to [Theorems 4.1–4.3](#) of [9]. When applied to this particular case, [Theorems 4.4](#) and [4.5](#) are new.

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